Harvard University
Computer Science 121

Problem Set 5

Due Tuesday, October 22, 2013 at 11:59 PM.
Submit your solutions electronically to cs121+ps5@seas.harvard.edu with "ps5 submission" in the subject line. The solutions to Parts A and B should be attached as separate PDF files, called lastname+ps5a.pdf and lastname+ps5b.pdf.
Late problem sets may be turned in until Friday, October 25, 2013 at 11:59 PM with a 20% penalty.
See syllabus for collaboration policy.

PART A (Graded by Gabe)

PROBLEM 1 (5+7 points)

Recall that given a function $f : \Sigma \to \Delta^*$, the function $h : \Sigma^* \to \Delta^*$ defined recursively by $h(\epsilon) = \epsilon$ and $h(w\sigma) = h(w)f(\sigma)$ is called a homomorphism.

(A) Show that if L is a context-free language over the alphabet Σ and $h : \Sigma^* \to \Delta^*$ is a homomorphism, then $h(L) = \{h(w) : w \in L\}$, is a context-free language over the alphabet Δ.

> **Solution:** Let $G = (V, \Sigma, R, S)$ be a grammar for L. Let g be the homomorphism $(V \cup \Sigma)^* \to (V \cup \Delta)^*$ extending the map $V \cup \Sigma \to (V \cup \Delta)^*$ that fixes V and sends $\sigma \in \Sigma$ to $f(\sigma)$. Let
>
> $$R' = \{(A \to g(u)) : A \to u \in R\}$$
>
> Then (V, Δ, R', S) is a grammar for $f(L)$ since $S \overset{*}{\Rightarrow}_G w$ if and only if $S \overset{*}{\Rightarrow}_{G'} f(w)$.

(B) Show that if L is a context-free language over the alphabet Δ and $h : \Sigma^* \to \Delta^*$ is a homomorphism, then $h^{-1}(L) = \{w : h(w) \in L\}$ is a context-free language over the alphabet Σ.

> **Solution:** Let $M = (Q, \Delta, \Gamma, \delta, q_0, F)$ be a PDA for L. For any letter $\sigma \in \Sigma$, let $Q_\sigma = \{q_\sigma^i : q \in Q, 0 < i \le |f(\sigma)|\}$. Let $M' = (Q \cup \bigcup_{\sigma \in \Sigma} Q_\sigma, \delta', q_0, F)$ where δ' is defined as follows: first, $\delta'(q, \sigma, \epsilon) = (q_\sigma^1, \epsilon)$; second, if $f(\sigma) = \alpha_1 \alpha_2 \cdots \alpha_n$ and $i < n$,
>
> $$\delta'(q_\sigma^i, \epsilon, \gamma) = \{(r_\sigma^i, \gamma') : (r, \gamma') \in \delta(q, \epsilon, \gamma)\} \cup \{(r_\sigma^{i+1}, \gamma') : (r, \gamma') \in \delta(q, \alpha_i, \epsilon)\}$$
>
> and similarly $\delta'(q_\sigma^n, \epsilon, \gamma) = \{(r_\sigma^n, \gamma') : (r, \gamma') \in \delta(q, \epsilon, \gamma)\} \cup \{(r, \gamma') : (r, \gamma') \in \delta(q, \alpha_n, \epsilon)\}$.
>
> Then M' accepts $f^{-1}(L)$: observe that for any $q, r \in Q$ there is a path in M' on $w \in \Sigma^*$ between q and r that takes the stack from $s \in \Gamma^*$ to $t \in \Gamma^*$ if and only if there is a path in M on $f(w)$ between q and r that takes the stack from s to t.

PROBLEM 2 (10 points)

Given a PDA M with the symbol \$ in its stack alphabet, let $\mathcal{N}(M)$ be the set of strings w such that there is a path through M on w starting at q_0 with the symbol \$ on the stack and ending with an empty stack. Show that a language L is context free if and only if there is a PDA M such that $\mathcal{N}(M) = L$.

Solution: Given a PDA $M = (Q, \Sigma, \Gamma, \delta, q_0, F)$, let $M_\sharp = (\{q_\sharp\} \cup Q \cup Q', \Sigma, \Gamma \cup \Gamma', \delta_\sharp, q_\sharp, Q')$ be such that

$$\begin{aligned}
\delta_\sharp(q_\sharp, \epsilon, \epsilon) &= \{(q_0, \$')\} \\
\delta_\sharp(q, \sigma, \gamma') &= \{(r', \epsilon) : (r, \epsilon) \in \delta(q, \sigma, \gamma)\} \cup \{(r, \xi') : (r, \xi) \in \delta(q, \sigma, \gamma)\} \\
\delta_\sharp(q', \sigma, \epsilon) &= \{(r, \gamma') : (r, \gamma) \in \delta(q, \sigma, \epsilon)\} \cup \{(r', \epsilon) : (r, \epsilon) \in \delta(q, \sigma, \epsilon)\} \\
\delta_\sharp(q, \sigma, \gamma) &= \delta(q, \sigma, \gamma)
\end{aligned}$$

Then $\mathcal{L}(M_\sharp) = \mathcal{N}(M)$ since for all $w \neq \epsilon$,

$$\delta_\sharp(q_\sharp, w, \epsilon) = \{(r, v\gamma') : (r, v\gamma) \in \delta(q_0, w, \$)\} \cup \{(r', \epsilon) : (r, \epsilon) \in \delta(q_0, w, \$)\}$$

Given a PDA M, assume WLOG that \$ is not in the stack alphabet of M. Let M' be M but with \$ in the stack alphabet and ϵ transitions from every final state to a new state where everything is stripped off the stack. Then $\mathcal{N}(M') = \mathcal{L}(M)$.

PROBLEM 3 (10 points)

Show that if A is context free and B is regular then the language $\{w : wx \in A \text{ for some } x \in B\}$ is context free.

Solution: Let $M = (Q, \Sigma, \Gamma, \delta, q_0, F)$ be a PDA for A and N a DFA for B. Let $M \times N$ be a the cross product of the machines M and N with each transition $q \overset{\sigma,\gamma\to\gamma'}{\longrightarrow} r$ replaced by the transition $q \overset{\epsilon,\gamma\to\gamma'}{\longrightarrow} r$. Then let M' be the (disjoint) union of M and $M \times N$ with an $(\epsilon, \epsilon \to \epsilon)$ transition from each r in M to $(r, q_0) \in M \times N$. Then M' accepts $\{w : wx \in A \text{ for some } x \in B\}$.