Harvard CS 121 and CSCI E-121

Lecture 24: Cook–Levin Theorem & More on NP

Harry Lewis

November 26, 2013

Proof of Cook-Levin Theorem: SAT is NP-complete

- Already know SAT $\in$ NP, so only need to show SAT is NP-hard.
- Let L be any language in NP. Let M be a NTM that decides L in time n^k. We define a polynomial-time reduction

$$f_L : \text{inputs} \mapsto \text{formulas}$$

such that for every w,

$$M \text{ accepts input } w \text{ iff } f_L(w) \text{ is satisfiable}$$

- Suppose M has state set Q and tape alphabet Γ
- M, Q, Γ are fixed; w varies!

Reduction via "computation histories"

Proof Idea: satisfying assignments of $f_L(w) \leftrightarrow$ accepting computations of M on w

Describe computations of M by boolean variables:

- If $n = |w|$, then any computation of M on w has at most n^k configurations, each of which is a string of length $\leq n^k$.
- Each configuration is a string of members of $C = Q \cup \Gamma \cup \{\#\}$, with exactly one member of Q (mark left and right ends with $\{\#\}$).

$\rightsquigarrow$ computation depicted by $n^k \times n^k$ "tableau" of members of C.

- Represent contents of cell (i, j) by $|C|$ boolean variables $\{x_{i,j,s} : s \in C\}$, where $x_{i,j,s} = 1$ means "cell (i, j) contains s".
- $0 \leq i, j < n^k$, so $|C| \cdot n^{2k}$ boolean variables in all

Subformulas that verify the computation

Express conditions for an accepting computation on w by boolean formulas:

- $\phi_{\text{cell}} =$
  "for each (i, j), there is exactly one $s \in C$ such that $x_{i,j,s} = 1$".
  A conjunction of n^{2k} subformulas.

- $\phi_{\text{start}} =$ "first row equals start configuration on w"

- $\phi_{\text{accept}} =$ "last row is an accept configuration on w"

- $\phi_{\text{move}} =$ "every 2×3 window is consistent with the transition function of M"

Completing the proof

Claim: Each of above can be expressed by a formula of size of size $O((n^k)^2) = O(n^{2k})$, and can be constructed in polynomial time from w.

Claim: M has an accepting computation on w if and only if $f_L(w) = \phi_{\text{cell}} \wedge \phi_{\text{start}} \wedge \phi_{\text{accept}} \wedge \phi_{\text{move}}$ has a satisfying assignment.

Thus $w \mapsto f_L(w)$ is a polynomial-time reduction from L to SAT.

Since above holds for every $L \in$ NP, SAT is NP-hard, as desired. ■

Clay Mathematics Institute

Dedicated to increasing and disseminating mathematical knowledge

P vs NP Problem

Suppose that you are organizing housing accommodations for a group of four hundred university students. Space is limited and only one hundred of the students will receive places in the dormitory. To complicate matters, the Dean has provided you with a list of pairs of incompatible students, and requested that no pair from this list appear in your final choice. This is an example of what computer scientists call an NP-problem, since it is easy to check if a given choice of one hundred students proposed by a coworker is satisfactory (i.e., no pair taken from your coworker's list also appears on the list from the Dean's office), however the task of generating such a list from scratch seems to be so hard as to be completely impractical. Indeed, the total number of ways of choosing one hundred students from the four hundred applicants is greater than the number of atoms in the known universe! Thus no future civilization could ever hope to build a supercomputer capable of solving the problem by brute force; that is, by checking every possible combination of 100 students. However, this apparent difficulty may only reflect the lack of ingenuity of your programmer. In fact, one of the outstanding problems in computer science is determining whether questions exist whose answer can be quickly checked, but which require an impossibly long time to solve by any direct procedure. Problems like the one listed above certainly seem to be of this kind, but so far no one has managed to prove that any of them really are so hard as they appear, i.e., that there really is no feasible way to generate an answer with the help of a computer. Stephen Cook and Leonid Levin formulated the P (i.e., easy to find) versus NP (i.e., easy to check) problem independently in 1971.

- The Millennium Problems
- Official Problem Description — Stephen Cook
- Lecture by Vijaya Ramachandran at University of Texas (video)
- Minesweeper

A Proof That P Is Not Equal To NP?

AUGUST 8, 2010

by rjlipton *tags:* P=NP, Proc

A serious proof that claims to have resolved the P=NP question.

Vinay Deolalikar is a Principal Research Scientist at HP Labs who has done important research in various areas of networks. He also has worked on complexity theory, including previous work on **infinite** versions of the P=NP question. He has just claimed that he has a proof that P is not equal to NP. That's right: $P \neq NP$. No infinite version. The real deal.

Today I will talk about his paper. So far I have only had a chance to glance at the paper; I will look at it more carefully in the future. I do not know what to think right now, but I am certainly hopeful.

The Paper

Fatal Flaws in Deolalikar's Proof?

AUGUST 12, 2010

by rjlipton

tags: finite model theory, flaws, Immerman, P≠NP, Proof

Possible fatal flaws in the finite model part of Deolalikar's proof

Neil Immerman is one of the world's experts on Finite Model Theory. He used insights from this area to co-discover the great **result** that NLOG is closed under complement.

Today I had planned not to discuss the proof, but I just received a note from Neil on Vinay Deolalikar "proof" that P≠NP. Neil points out two flaws in the finite model part that sound extremely damaging to me. He has already shared them with Vinay, and suggested that I highlight them here. The comments from Neil are in the next section—I have only edited it slightly to make it "compile."

Deolalikar's Claim: One Year Later

AUGUST 11, 2011

by rjlipton

tags: claim, Deolalikar, Proof

Still no final chapter, lessons learned

Vinay Deolalikar just over a year ago claimed that he had a proof that $P \neq NP$.

Today I want to make a short comment on the status of his claim.

I recently received an email asking: Has the anniversary slipped your notice?

No. But there is not a lot to say about the situation—for better or for worse. Here is all that we know publicly:

- His web **page** still claims the result, and explains that it is out to a journal for the standard refereeing process.
- A few weeks ago when I inquired "what is up with the proof," his e-mail reply said the same thing. He still believes that he has a proof, and reiterated that it is being checked by referees.
- He has expanded and updated the paper, and claims to have answered all the issues that were raised on the web here and elsewhere.
- He will not let the public see his proof, nor will he post it, so it remains an unknown.

Today: paper is "Out for refereeing"

> Vinay Deolalikar. P ≠ NP. *The preliminary version was meant to solicit feedback from a few researchers as is customarily done. It illustrated the interplay of principles from various areas, which was the major effort in constructing the proof. I have fixed all the issues that were raised about the preliminary version in a revised manuscript; clarified some concepts; and obtained simpler proofs of several claims. Once I hear back from the journal as part of due process, I will put up the final version on this website.*

Gasarch's Community Poll

Most professionals think P $\neq$ NP but the question won't be resolved soon.

Richard Karp: (Berkeley, unsure, P$\neq$NP) My intuitive belief is that P is unequal to NP, but the only supporting arguments I can offer are the failure of all efforts to place specific NP-complete problems in P by constructing polynomial-time algorithms.

I believe that the traditional proof techniques will not suffice. Something entirely novel will be required.

My hunch is that the problem will be solved by a young researcher who is not encumbered by too much conventional wisdom about how to attack the problem.

co-NP

Recall that co-NP $= \{\overline{L} : L \in \text{NP}\}$.

Some co-NP-complete problems:

- Complement of any NP-complete problem.
- TAUTOLOGY = $\{\varphi : \forall a\ \varphi(a) = 1\}$ (even for 3-DNF formulas φ).

Believed that NP $\neq$ co-NP, P $\neq$ NP $\cap$ co-NP.

Between P and NP-complete

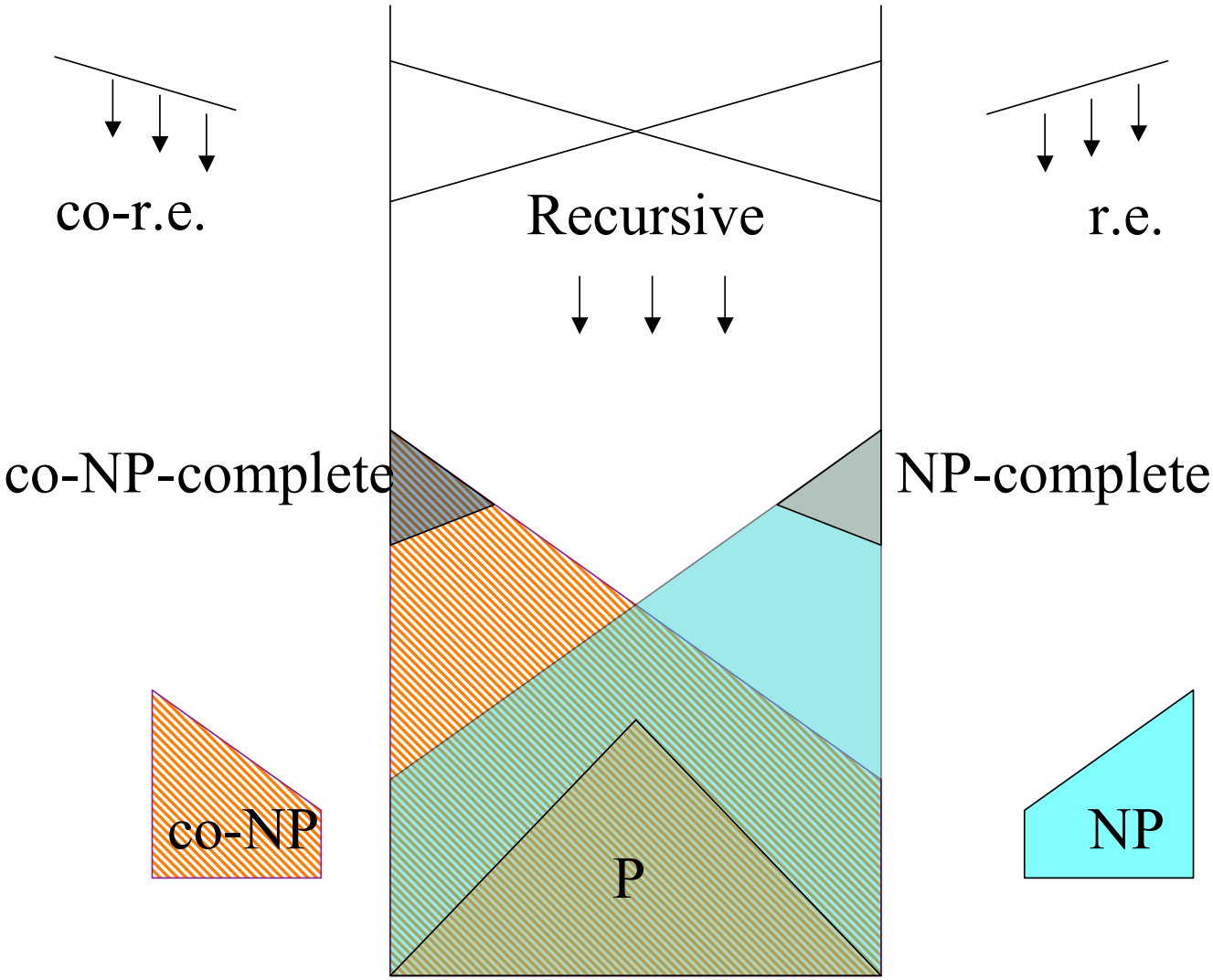
Theorem: If P $\neq$ NP, then there are NP languages that are neither in P nor NP-complete.

Proof: beyond the scope of this course.

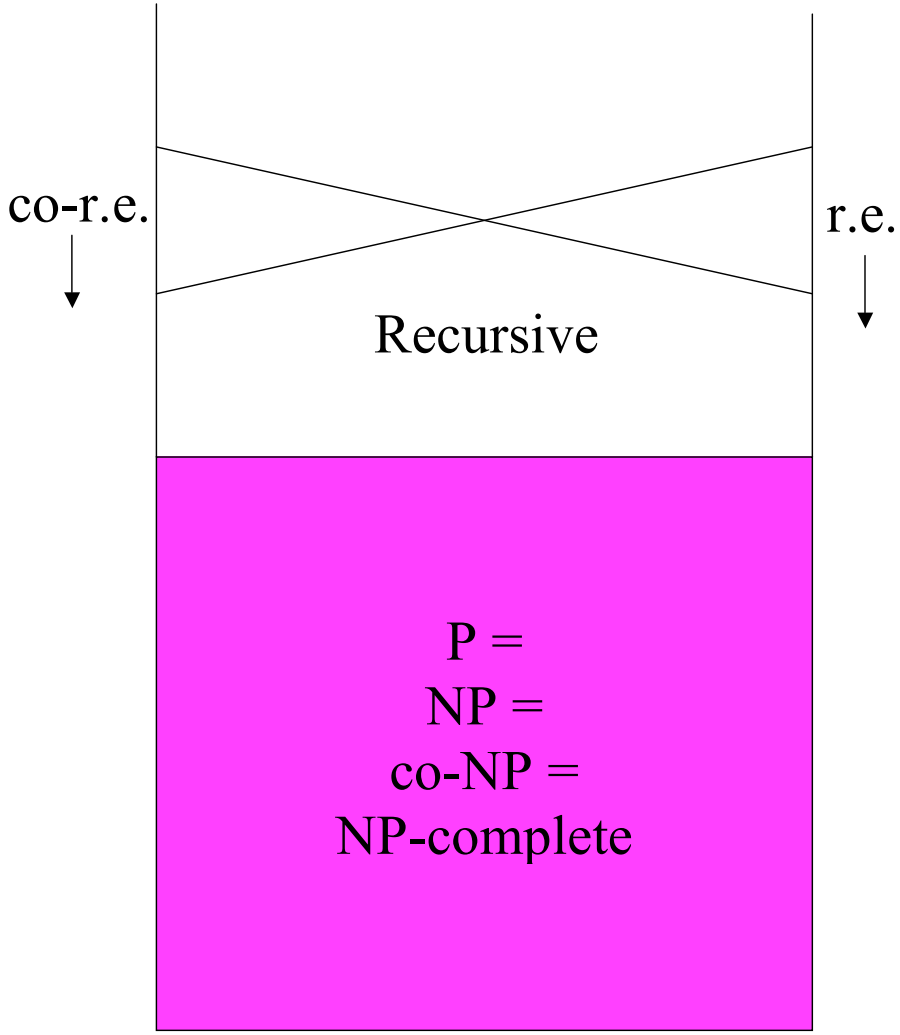
Some natural candidates:

- GRAPH ISOMORPHISM
- Any problem in NP $\cap$ co-NP for which we don't know a poly-time algorithm.
- E.g. FACTORING $= \{\langle m, n\rangle : n$ has a factor $\leq m\}$
- Why is FACTORING $\in$ NP $\cap$ co-NP?

The World If P ≠ NP



The World If P = NP



Beyond NP

- "Space" as a resource $=$ number of TM squares used in a computation
- A reasonable proxy for memory on other computational models
- PSPACE $=$ languages decidable on TMs using polynomial space
- P $\subseteq$ PSPACE (why?)
- NP $\subseteq$ PSPACE (why?)
- PSPACE $\subseteq$ NPSPACE (why?)

Examples of problems in PSPACE or NPSPACE but probably not in NP

- Determining whether $w \in L(G)$, where G is a context-sensitive grammar, is in NPSPACE
  - Recall that G is a CSG if the right-hand side of every rule is at least as long as its left-had side
  - I.e. if $u \rightarrow v$ is a rule then $|u| \leq |v|$
- Determining whether a quantified boolean expression is true is in PSPACE
  - E.g. $\forall x \exists y \forall z [(x \wedge y) \vee (\neg x \wedge \neg z)]$

Savitch's Theorem

NPSPACE $=$ PSPACE

- How much time can a computation take if it uses $O(n^k)$ space and does not loop? $O(2^{n^k})$
- To check deterministically if there exists a computation from TM configuration C_1 to TM configuration C_2 in T steps,
  - for all configurations C, check if
    - there is a computation from C_1 to C of $T/2$ steps and
    - there is a computation from C to C_2 in $T/2$ steps
- To check whether initial configuration yields final configuration takes $O(\log(2^{n^k})) = O(n^k)$ recursion levels and $O(n^k)$ space at each level $\Rightarrow O(n^{2k})$ space

LOGSPACE

A problem is in LOGSPACE if it is decided by a TM that uses only logarithmic space on a work tape and doesn't write on the input tape.

$$\text{LOGSPACE} \subseteq \text{P} \subseteq \text{NP} \subseteq \text{PSPACE} = \text{NPSPACE}$$

LOGSPACE $\subsetneq$ PSPACE by a diagonalization argument

So at least one of the three $\subseteq$s must be a $\subsetneq$ but we don't know which!