# Harvard CS 121 and CSCI E-121

# Lecture 15: Recognizability & Decidability

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- Reading: Sipser §3.2,§3.3,§4.1.

# Nondeterministic TMs

- Like TMs, but $\delta : Q \times \Gamma \rightarrow P(Q \times \Gamma \times \{L, R\})$
- It mainly makes sense to think of NTMs as recognizers

  $L(M) = \{w : M$ has some accepting computation on input $w\}$

  **Example:** NTM to recognize
  $\{w : w$ is the binary notation for a product of two integers $\geq 2\}$

# NTMs recognize the same languages as TMs

- Given a NTM $M$, we must construct a TM $M'$ that determines, on input $w$, whether $M$ has an accepting computation on input $w$.

- $M'$ systematically tries

  $\rightarrow$ all one-step computations

  $\rightarrow$ all two-step computations

  $\rightarrow$ all three-step computations

  ⋮

# Enumerating Computations by Dovetailing

- There is a bounded number of $k$-step computations, for each $k$.

  (because for each configuration there is only a constant number of "next" configurations in one step)

- Ultimately $M'$ either:
  - discovers an accepting computation of $M$, and accepts itself,
  - or searches forever, and does not halt.

# Dovetailing Details

- Suppose that the maximum number of different transitions for a given $(q, a)$ is $C$.
- Number those transitions $1, \ldots, C$ (or less)
- Any computation of $k$ steps is determined by a sequence of $k$ numbers $\leq C$ (the "nondeterministic choices").
- How $M'$ works: 3 tapes

$\#1$ | Original input to $M$ $\sqcup$

$\#2$ | Simulated tape of $M$

$\#3$ | $1213\sqcup\cdots$ Nondeterministc choices for $M'$

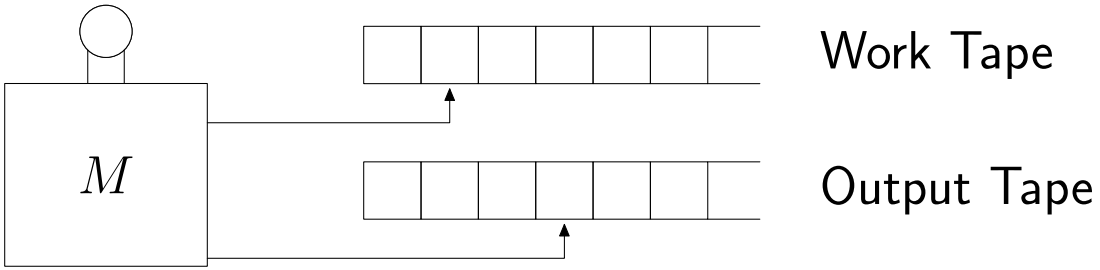
# Simulating one step of $M$

- Each major phase of the simulation by $M'$ is to simulate one finite computation by $M$, using tape #3 to resolve nondeterministic ambiguities.
- Between major phases, $M'$
  - erases tape #2 and copies tape #1 to tape #2
  - Replaces string in $\{1, \ldots, C\}^*$ on tape #3 with the lexicographically next string to generate the next set of nondeterministic choices to follow.
- Claim: $L(M') = L(M)$
- **Q:** Slowdown?

# Another TM Variant: Enumerators

**Def**: A TM $M$ enumerates a language $L$ if $M$, when started from a blank tape, runs forever and "emits" all and only the strings in $L$.

(For example, by writing the string on a special tape and passing through a designated state.)



# Recognizable $\equiv$ enumerable

**Theorem**: $L$ is Turing-recognizable iff $L$ is enumerated by some TM.

**Proof:**

($\Rightarrow$) Suppose $L(M) = L$. We want to construct a TM $M'$ that enumerates $L$.

$M'$ dovetails all of the computations by $M$:

1. Do 1 step of $M$'s computation on $w_0$
2. Do 2 steps of $M$ on $w_0$ and $w_1$
3. Do 3 steps on each of $w_0, w_1, w_2$

where $w_0, w_1, \ldots =$ lexicographic enumeration of $\Sigma^*$.

Outputting any strings $w_i$ whose computations have accepted.

# Recognizable $\equiv$ enumerable, finis

$(\Leftarrow)$

- The Turing-decidable sets are often called *recursive* because they can be computed using certain systems of recursive equations, rather than via TMs.
- The Turing-recognizable sets are usually called *recursively enumerable*, i.e. "computably enumerable," due to the above characterization in terms of enumerators.
- **Fact:** $L$ is decidable iff it is enumerable in *lexicographic order*.

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   Proof:
   Switch $q_{\text{accept}}$ and $q_{\text{reject}}$.

# Conclusion about Recursive and R.E. Sets

3. $L$ is recursive if and only if both $L$ and $\overline{L}$ are r.e.

   Proof: . . .

# Asking questions about arbitrary finite automata

- **Proposition:** Every regular language is decidable.

  **Proof:** (By “coding” a DFA as a TM.)

# What if the DFA $D$ is part of the input?

- That is, can we design a single TM that, given two inputs, $D$ and $w$, decides whether $D$ accepts $w$?
  - The TM needs to use a fixed alphabet & state set for all inputs $D, w$.
- **Q:** How to represent $D = (Q, \Sigma_D, \delta, q_0, F)$ and $w$?
  List each component of the 5-tuple, separated by |'s.
  - Represent elements of $Q$ as binary strings over $\{0, 1\}$, separated by ,'s.
  - Represent elements of $\Sigma_D$ as binary strings over $\{0, 1\}$, separated by ,'s.
  - Represent $\delta : Q \times \Sigma_D \to Q$ as a sequence of triples $(q, \sigma, q')$, separated by ,'s, etc.

  We denote the encoding of $D$ and $w$ as $\langle D, w \rangle$.

# A "Universal" algorithm for deciding regular languages

- **Proposition:** $A_{\text{DFA}} = \{\langle D, w\rangle : D \text{ a DFA that accepts } w\}$ is decidable.

  **Proof sketch:**

  - First check that input is of proper form.
  - Then simulate $D$ on $w$. Implementation on a multitape TM:
    - Tape 2: String $w$ with head at current position (or to be precise, its representation).
    - Tape 3: Current state $q$ of $D$ (i.e., its representation).

- Could work with other encodings, e.g. transition function as a matrix rather than list of triples.

# Representation independence

**General point:** Notions of computability (e.g. decidability and recognizability) are independent of data representations.

- A TM can convert any reasonable encoding to any other reasonable encoding.
- We will use $\langle \cdot \rangle$ to mean "any reasonable encoding".
- We'll need to revisit representation issues again when we discuss computational *speed.*
- For the moment when we are interested only in whether problems are decidable, undecidable, recognizable, etc., so we can be content knowing that there is *some* representation on which an algorithm could work.