CS 121 Midterm Solutions

October 17, 2013

Answer all questions in your exam booklet, not on this sheet. There are more problems on the back of the page. 75 points total. $\Sigma = \{a, b\}$ unless otherwise specified.

PROBLEM 1 (12 points)

True or False (1 point each, no explanations).

(a) $\emptyset \subseteq \{\emptyset, a, b\}$

(b) $\emptyset \in \{\emptyset, a, b\}$

(c) $|\mathcal{P}(\emptyset)| = 0$.

(d) For any two languages L_1 and L_2, if $L_1 \cup L_2$ is regular, then L_1 and L_2 are regular.

(e) For any two languages L_1 and L_2, $L_1^* \circ L_2^* \subseteq (L_1 \cup L_2)^*$.

(f) If L_1 and L_2 are *finite* languages, then $|L_1 \circ L_2| = |L_1| \cdot |L_2|$. ("$\cdot$" denotes multiplication.)

(g) If L is regular and L' is not regular, then the language $L \circ L'$ is not regular.

(h) If M is any NFA with n states, then there is a DFA for $\mathcal{L}(M)$ with at most n^2 states.

(i) Every non-regular language is a union of countably many regular languages.

(j) Every language is either finite or cofinite.

(k) There are countably many context-free languages.

(l) $\{a^n w b^n : n \geq 0, w \in \Sigma^*\}$ is context-free but not regular.

Solution: (a) T (b) T (c) F (d) F (e) T (f) F (g) F (h) F (i) T (j) F (k) T (l) F

PROBLEM 2 (10 points)

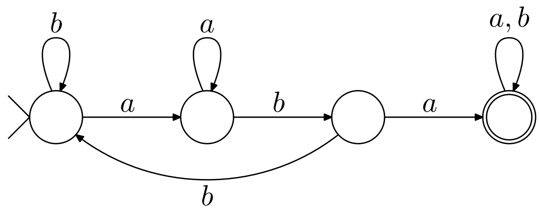
Consider the following binary relation on sets: X COVERS Y if there is an surjective (onto) function from X to Y. Prove that COVERS is reflexive but not symmetric.

Solution: COVERS is reflexive. Given any set X, the identity function $id : X \to X$ where $id(x) = x$ for all $x \in S$ is an onto function from X to itself. COVERS is not symmetric. The function $f : \mathbb{N} \to \{1\}$ defined by $f(x) = 1$ for all $x \in \mathbb{N}$ is onto, but there is no onto function from $\{1\}$ to $\mathbb{N}$.

PROBLEM 3 (5 points)

Construct a DFA for the language $L = \{w : w \text{ contains the substring } aba\}$. You need not prove that your construction is correct.

Solution:



PROBLEM 4 (5 points)

String s is a **subsequence** of string w iff the symbols of s appear in the same order in w, but not necessarily consecutively. Write a regular expression for $L = \{w : aba \text{ or } bb \text{ is a subsequence of } w\}$

Solution: $(\Sigma^* a \Sigma^* b \Sigma^* a \Sigma^*) \cup (\Sigma^* b \Sigma^* b \Sigma^*)$

PROBLEM 5 (10 points)

Explain how to determine, given a DFA M, whether $\mathcal{L}(M) = \Sigma^*$.

Solution: A DFA M accepts Σ^* if and only if all the reachable states of M are accept states. Another acceptable answer is that M accepts Σ^* if and only if the result of minimizing M is a DFA isomorphic to the minimal DFA for Σ^*. An unacceptable answer converts M to a regular expression R and then claims without proof that it's any easier to figure out if R generates Σ^*.

PROBLEM 6 (8 points)

Show that if L_1 and L_2 are regular languages, then $L = \{w : w \in L_1 \text{ or } w \in L_2 \text{ but not both}\}$ is also a regular language.

Solution:

$$\begin{aligned} L &= (L_1 \cup L_2) \setminus (L_1 \cap L_2) \\ &= (L_1 \cup L_2) \cap (L_1 \cap L_2)^c \end{aligned}$$

It follows from the closure of regular languages under union and intersection that $L_1 \cup L_2$ and $L_1 \cap L_2$ are regular, so that the desired result follows from the closure of regular languages under intersection and complement.

PROBLEM 7 (5 + 10 points)

(a) Show that the set of strings with exactly two a's is regular by presenting an appropriate DFA or regular expression (no proof needed).

(b) Prove that the set of all strings with exactly two more a's than b's is not regular.

Solution:

(a) $b^*ab^*ab^*$

(b) Let L be the set of all strings with exactly two more a's than b's; suppose for the sake of contradiction that it is regular. Then, it has some pumping length p. Consider the string $a^{p+2}b^p$; since it has length greater than p, we know it can be pumped. However, by the conditions of the pumping lemma, if we write $a^{p+2}b^p = xyz$, then $|xy| \leq p$ means that x and y consist of only a's. Consequently, xy^2z contains $p + 2 + |y|$ a's but only p b's, so it is outside the language, a contradiction.

PROBLEM 8 (5 + 5 points)

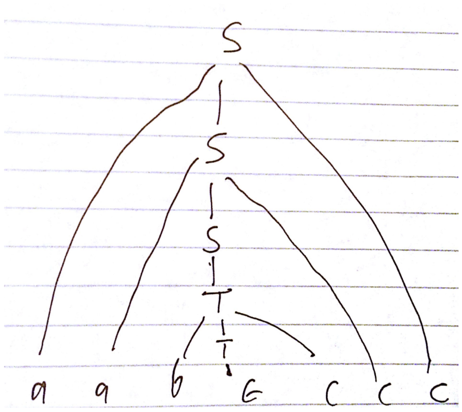
Consider the context-free grammar $G = (\{S, T\}, \{a, b, c\}, R, S)$, where R is the following set of rules:

$$S \to aSc \mid T$$
$$T \to bTc \mid \varepsilon$$

(a) Draw the parse tree for $aabccc$.

(b) Describe $\mathcal{L}(G)$ simply.

Solution:

(a) 

(b) The language with the same number of a's and b's together as the number of c's in alphabetical order. More formally:

$$\mathcal{L}(G) = \{a^m b^n c^{m+n} : m, n \geq 0\}$$

THE END