Harvard CS121 and CSCI E-121

Lecture 2: Mathematical Preliminaries

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Reading: Sipser, Chapter 0

Sets

- **Sets** are defined by their members

  $A = B$ means that for every x, $x \in A$ iff $x \in B$

  Example: $\mathcal{N} = \{0, 1, 2, \ldots\}$

- **Cardinality**

  Sets can be **finite** (e.g. $\{1, 3, 5\}$) or **infinite** (e.g. $\mathcal{N}$).

  Q: Is $\{\mathcal{N}\}$ finite?

  If A is finite ($A = \{a_1, \ldots, a_n\}$ for some $n \in \mathcal{N}$), then its **cardinality** (or **size**) $|A|$ is n, the number of elements in A.

  The **empty set** $\emptyset$ has cardinality 0.

  Cardinality of infinite sets to be discussed later!

Set Operations

$\cup$	union	$\{a,b\} \cup \{b,c\} = \{a,b,c\}$
$\cap$	intersection	$\{a,b\} \cap \{b,c\} = \{b\}$
$-$	difference	$\{a,b\} - \{b,c\} = ??$
$\setminus$	another name for $-$	

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  e.g. $P(\{a,b\}) = \{\emptyset, \{a\}, \{b\}, \{a,b\}\}$

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- The **Cartesian product** of sets A, B

  $A \times B = \{(a,b) : a \in A, b \in B\}$

  triples, . . .

Some little exercises

- Assume all sets are finite
- What is the relation between $|A|$, $|B|$, and $|A \cup B|$?
- $|A - B|$?
- $|A \times B|$?
- $|P(S)|$?

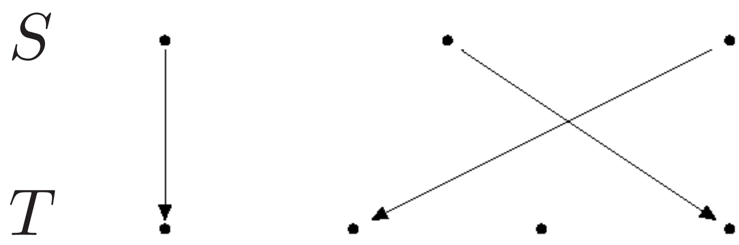
Functions

A **function** $f : S \rightarrow T$ maps each element $s \in S$ to (exactly one) element of T, denoted $f(s)$.

For example, $f(n) = n^2$ is a function from $\mathcal{Z} \rightarrow \mathcal{N}$

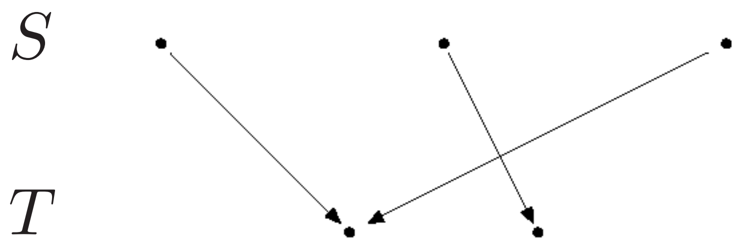
($\mathcal{Z} =$ all integers)

Special varieties of functions



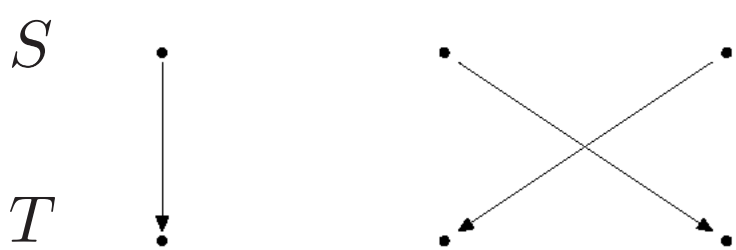
<u>1-1:</u>

$s_1 \neq s_2 \Rightarrow$
$f(s_1) \neq f(s_2)$



<u>Onto:</u>

For every $t \in T$
there is an $s \in S$
such that $f(s) = t$



<u>Bijection:</u>

1-1 and onto
“1-1 Correspondence”

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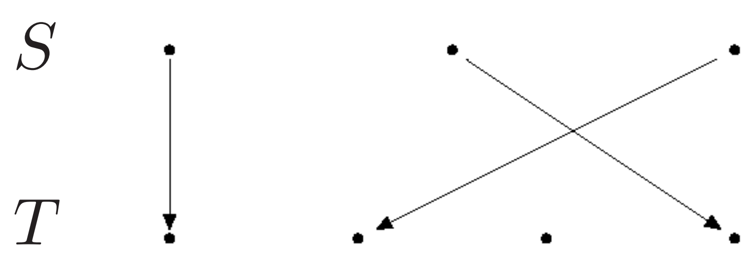

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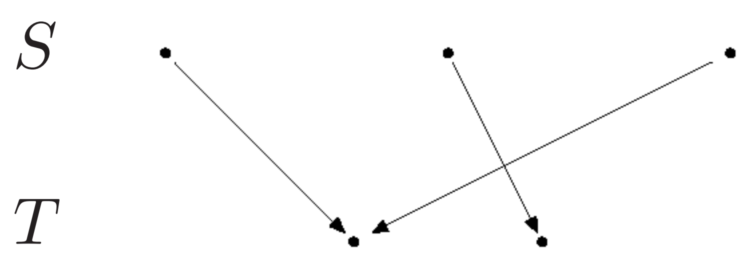

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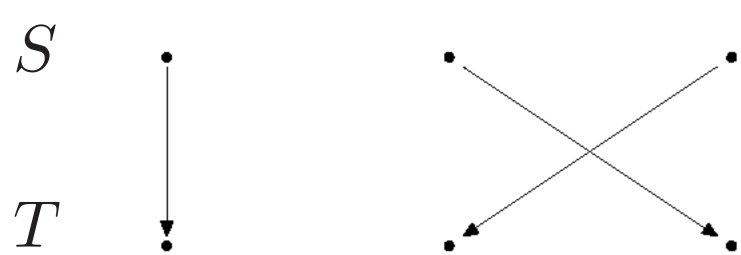


Bijection:
1-1 and onto
"1-1 Correspondence"

- Formal definition of cardinality: S has (finite) cardinality $n \in \mathcal{N}$ iff there is a bijection $f : \{1, \ldots, n\} \to S$.

Relations

- A k-ary **relation** on $S_1, \ldots, S_k$ is a subset of $S_1 \times \ldots \times S_k$

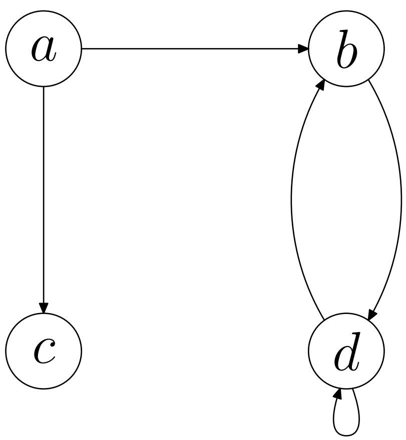
  [A function $f : S \rightarrow T$ corresponds to the relation $\{(s, f(s)) : s \in S\} \subseteq S \times T$.]

- A **binary relation** on S is a subset of $S \times S$

- For example, $(\mathrm{GWB}, \mathrm{GHWB}) \in \mathrm{Son}$, where $\mathrm{Son} = \{(x, y) : x \text{ is a son of } y\}$

What is a (directed) graph?

- A binary relation can be pictured as a “directed graph”:



- (Still assuming a finite set of nodes)
- Every edge has a direction
- Only one arrow between two nodes in the same direction
- Self-loops possible

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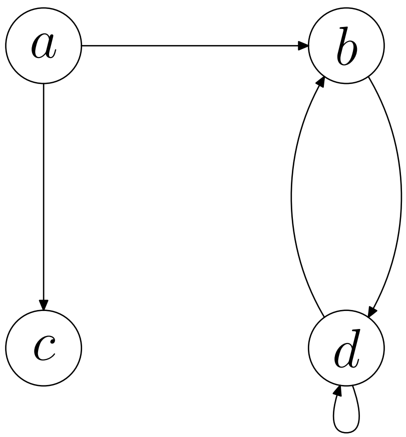

$$= \quad \{(a,b),(a,c),(b,d),(d,b),(d,d)\}$$

- Formally, a **directed graph** G consists of a finite set V of **vertices** (or **nodes**), and a set of edges $E \subseteq V \times V$.

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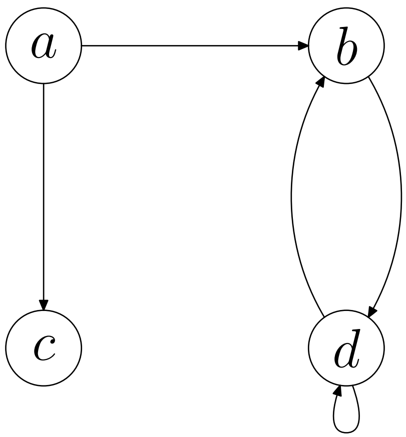

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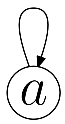
- Formally, a **directed graph** G consists of a finite set V of **vertices** (or **nodes**), and a set of edges $E \subseteq V \times V$.

- **NB**: Because a relation (or edge set) is a *set* (of ordered pairs), there can be only one arrow from one node to another.

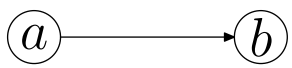
Properties of Binary Relations

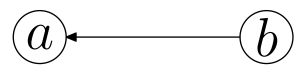
A relation $R \subseteq S \times S$ is:

- **reflexive** if 

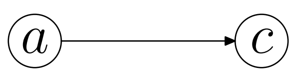
for each $a \in S$

  i.e., $(a, a) \in R$ for each $a \in S$

- **symmetric** if 

whenever 

  i.e., for any $a, b \in S$, if $(a, b) \in R$ then $(b, a) \in R$

- **transitive** if 

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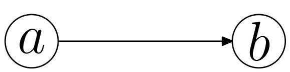
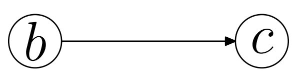
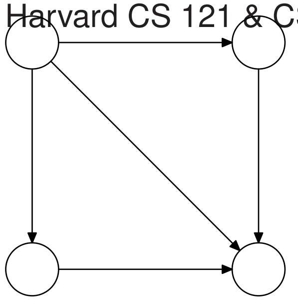
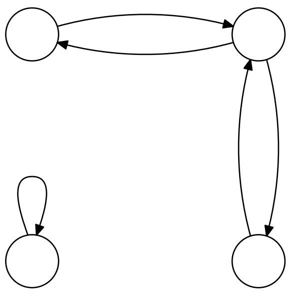
  

and 

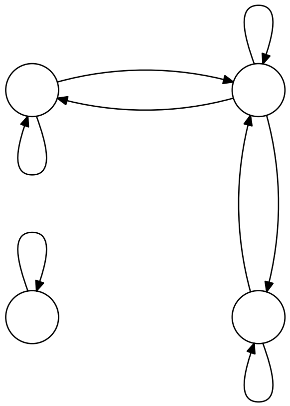
  i.e., if $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$



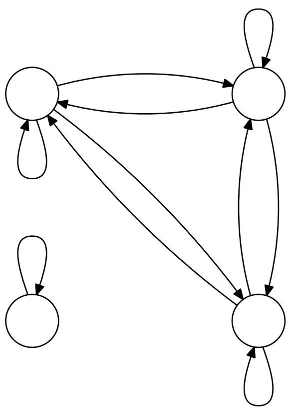
Transitive, not symmetric nor reflexive



Symmetric, not reflexive, nor transitive



Reflexive & symmetric, not transitive



Reflexive, transitive, and symmetric

Equivalence Relations

A relation that satisfies all three properties is called an **equivalence relation**

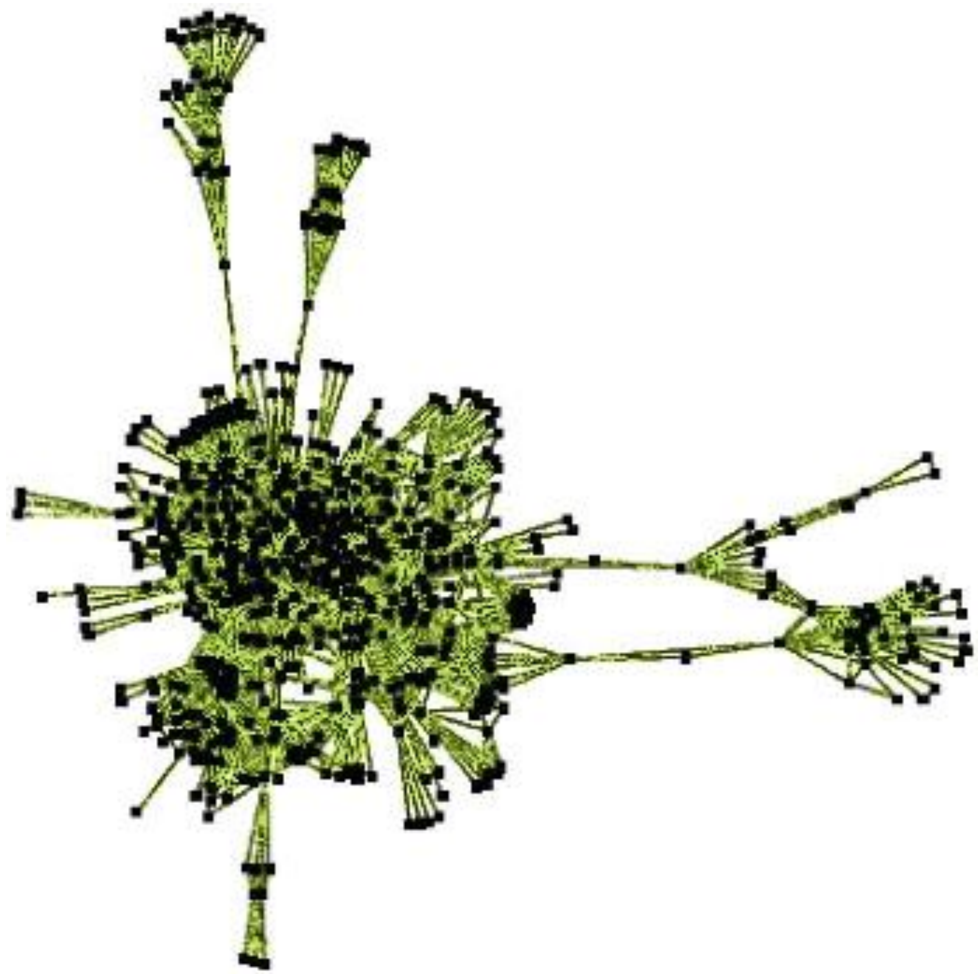
An equivalence relation decomposes S into **equivalence classes**—any two members of the same equivalence class bear the relation to each other.

Which Properties Do These Relations Have?

Domain	Relation
People	Lives-In-The-Same-City-As
People	Is-An-Ancestor-Of
People	Is-A-Brother-Of

Undirected graphs

A symmetric relation with no self-loops (that is, for every a, $(a, a) \notin R$) can be depicted as an *undirected graph*



Source: http://www.howweknowus.com/2008/07/23/great-work-lousy-title/

Subject: Re: six degrees to harry lewis

On Friday, January 23, 2004, at 05:09 AM, Mark Elliot Zuckerberg wrote:

Professor,

I've been interested in graph theory and its applications to social networks for a while now, so I did some research (on my own time) that has to do with linking people through articles they appear in from the Crimson.

I thought people would find this interesting, so I've set up a preliminary site that allows people to find the connection (through people and articles) from any person to the most frequently mentioned person in the time frame I looked at. This person is you.

Six degrees, continued

```
From:  Harry Lewis <lewis@deas.harvard.edu>
Date:  January 23, 2004 4:25:44 PM EST
To:  Mark Elliot Zuckerberg <mzuckerb@fas.harvard.edu>
Subject:  Re:  six degrees to harry lewis
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Six degrees, continued

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Sure, what the hell.
Seems harmless.

Strings and Languages

- **Symbol** a, b, ...

- **Alphabet** A finite, nonempty set of symbols

  usually denoted by Σ

- **String** (informal) Finite number of symbols “put together”

  e.g. $abba$, b, bb

  Empty string denoted by ε

- $\Sigma^* =$ set of all strings over alphabet Σ

  e.g. $\{a, b\}^* = \{\varepsilon, a, b, aa, ab, \ldots\}$

More on Strings

- **Length** of a string x is written $|x|$

$$|abba| = 4$$

$$|a| = 1$$

$$|\varepsilon| = 0$$

The set of strings of length n is denoted Σ^n.

Concatenation

- **Concatenation** of strings

  Written as $x \cdot y$, or just xy

  Just follow the symbols of x by the symbols of y

  $x = abba$, $y = b \Rightarrow xy = abbab$

  $x\varepsilon = \varepsilon x = x$ for any x

- The **reversal** x^R of a string x is x written backwards.

  If $x = x_1 x_2 \cdots x_n$, and each $x_i \in \Sigma$, then $x^R = x_n x_{n-1} \cdots x_1$.

Formal Inductive Definitions

- Like recursive data structures and recursive procedures when programming.

- **Strings** and their **length**:

  ε is a string of length 0.

  If x is a string of length n and $\sigma \in \Sigma$, then $x\sigma$ is a string of length $n + 1$.

  (i.e. start with ε and add one symbol at a time, like $\varepsilon aaba$, but we don't write the initial ε unless the string is empty)

Inductive definitions of string operations

- The **concatenation** of x and y, defined by induction on $|y|$.

  $[\,|y| = 0\,]$ $\quad x \cdot \varepsilon = x$

  $[\,|y| = n+1\,]$ write $y = z\sigma$ for some $|z| = n$, $\sigma \in \Sigma$
  define $x \cdot (z\sigma) = (x \cdot z)\sigma$

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- The **reversal** of x, defined by induction on $|x|$:

  $[\,|x| = 0\,]$ $\quad \varepsilon^R = \varepsilon$

  $[\,|x| = n+1\,]$ $\quad (y\sigma)^R = \sigma \cdot y^R$,
  for any $|y| = n$, $\sigma \in \Sigma$

An inductive proof

- Proposition: The n'th Fibonacci number is the number of strings over $\{a, b\}$ of length $n - 2$ with no consecutive b's.
- First step: make sure you know what everything means
  - What are the Fibonacci numbers?
  - Where does the indexing start?
  - How does the statement of the proposition need to be cleaned up to be precise?
- Is the proposition true for the smallest value of n?
- If it is true for small values of n, say for all $n \leq k$, why would it be true for $n = k + 1$?

So here is what we want to prove:

Proposition: For every $n \geq 2$, F_n is the number of strings over $\{a, b\}$ of length $n - 2$ with no consecutive b's.

- **Proof:**

  Base case: Prove it for $n = 2$ and $n = 3$, by a hand check.

  Induction step:

  - For any n, let X_n be the set of strings of length n with no consecutive b's. We have proven that $|X_n| = F_{n+2}$ for $n = 0$ and $n = 1$. Now let k be any number ≥ 1 and assume we have proven that $|X_n| = F_{n+2}$ for all $n \leq k$. We want to conclude that $|X_{k+1}| = F_{k+3}$.

- If $x \in X_{k+1}$ then x ends with either a or b. So let's split X_{k+1} into those two subsets, A_{k+1} and B_{k+1}, and count how many strings are in each.

- The number of strings in A_{k+1} is the number of strings in X_k since stripping off the final a leaves a string with no consecutive b's, and it could be *any* string of length k with no consecutive b's. But we already know that $|X_k| = F_{k+2}$.

- The number of strings in B_{k+1} is the number of strings in X_{k-1}, since a string of length $k+1$ ending in b must actually end in ab, and the rest of the string, dropping the last two symbols, could be any string of length $k-1$ with no consecutive b's. We already know that $|X_{k-1}| = F_{k+1}$.

- So $|X_{k+1}| = |A_{k+1}| + |B_{k+1}| = F_{k+2} + F_{k+1} = F_{k+3}$.

Languages

- A **language** L over alphabet Σ is a set of strings over Σ (i.e. $L \subseteq \Sigma^*$)

  Computational problem: given $x \in \Sigma^*$, is $x \in L$?

  Any YES/NO problems can be cast as a language.

- Examples of simple languages:
  - All words in the *American Heritage Dictionary* $\{a, aah, aardvark, \ldots, zyzzva\}$.
  - $\emptyset$
  - Σ^*
  - Σ
  - $\{x \in \Sigma^* : |x| = 3\} = \{aaa, aab, aba, abb, baa, bab, bba, bbb\}$

More complicated languages

- The set of strings $x \in \{a, b\}^*$ such that x has more a's than b's.
- The set of strings $x \in \{0, 1\}^*$ such that x is the binary representation of a prime number.
- All 'C' programs that do not go into an infinite loop.
- $L_1 \cup L_2$, $L_1 \cap L_2$, $L_1 - L_2$ if L_1 and L_2 are languages.

$\vdots$

The highly abstract and metaphorical term "language"

- A language can be either finite or infinite
- A language need not have any "internal structure"

Be careful to distinguish

ε The empty string (a string)

$\emptyset$ The empty set (a set, possibly a language)

$\{\varepsilon\}$ The set containing one element, which is the empty string (a language)

$\{\emptyset\}$ The set containing one element, which is the empty set (a set of sets, maybe a set of languages)

Operations on Languages

- Set operations $\cup$ $\cap$ $-$

- Concatenation of Languages

$$L_1 L_2 = \{xy : x \in L_1, y \in L_2\}$$

e.g. $\{a, b\}\{a, bb\} = \{aa, ba, abb, bbb\}$

e.g. $\{\varepsilon\}L = L$

e.g. $\emptyset L =$?

Kleene star

$$L^* = \{w_1 \cdots w_n : n \geq 0, w_1, \ldots, w_n \in L\}$$

e.g. $\{aa\}^* = \{\varepsilon, aa, aaaa, \ldots\}$

e.g. $\{ab, ba, aa, bb\}^* =$ all even length strings

e.g. $\Sigma^* =$ Kleene Star of Σ

e.g. $\emptyset^* =$?