Harvard University
Computer Science 121
Problem Set 0

Due Tuesday, September 10, 2013 at 11:59 PM.
Submit your solutions electronically to cs121+ps0@seas.harvard.edu with "ps0 submission" in the subject line. The solutions to Parts A and B should be attached as separate PDF files, called lastname+ps0a.pdf and lastname+ps0b.pdf.
See syllabus for collaboration policy.

Note: This is our standard prologue. Problem set 0 counts 0 points, but you should complete it, and we will grade it and provide a solution set for your edification.

PART A (Graded by Gabe)

PROBLEM 1 (2+2+2+2 points)

Using set notation, give formal descriptions of the following sets:
(A) The set containing no elements.
(B) The set containing the empty string.
(C) The power set of a set X, denoted $\mathcal{P}(X)$, i.e., the set containing all subsets of X.
(D) The difference between two sets X and Y, denoted $X \setminus Y$, i.e., the set containing all elements of X that are not elements of Y.
Solution.
(A) $\{x : x \neq x\}$.
(B) $\{\varepsilon\}$.
(C) $\{Y : Y \subseteq X\}$.
(D) $\{x : x \in X \land x \notin Y\}$.

PROBLEM 2 (10+10 points)

Given a set X, we define the power set $\mathcal{P}(X)$ to be the set of all subsets of X.
(A) Construct a bijection between $\mathcal{P}(X)$ and the set of functions from X into the set $\{0, 1\}$.
(B) Prove that for any finite set X, $|P(X)| = 2^{|X|}$. (Hint: use induction on $|X|$.)
Solution.
(A) For a subset Y of X, define $f(Y) = \chi_Y$ where χ_Y is the characteristic function of the set Y, that is,

$$\chi_Y(x) = \begin{cases} 1 & x \in Y \\ 0 & x \notin Y \end{cases}$$

The mapping f is obviously invertible, i.e., it is a bijection.

(B) Proceeding by induction, we note that the only subset of the empty set is itself. Therefore $|\mathcal{P}(\emptyset)| = 1 = 2^0$, which establishes the base case. Assume inductively that for any set Y of size i, $|\mathcal{P}(Y)| = 2^i$. Let Z be a set of size $i+1$. Let z be an element of Z. Let Z_0 and Z_1 be, respectively, the set of subsets that contain z and the set of subsets that do not contain z. It follows that $Z_1 = \mathcal{P}(Z - z)$, and there is an obvious bijection between Z_0 and $\mathcal{P}(Z - z)$. It follows from the induction hypothesis that $|Z_1| = |Z_0| = 2^i$. It follows that $|\mathcal{P}(Z)| = |\mathcal{P}(Z_0)| + |\mathcal{P}(Z_1)| = 2^i + 2^i = 2^{i+1}$.