Harvard University
Computer Science 121

Problem Set 0

Due Tuesday, September 10, 2013 at 11:59 PM.
Submit your solutions electronically to cs121+ps0@seas.harvard.edu with "ps0 submission" in the subject line. The solutions to Parts A and B should be attached as separate PDF files, called lastname+ps0a.pdf and lastname+ps0b.pdf.
See syllabus for collaboration policy.

Note: This is our standard prologue. Problem set 0 counts 0 points, but you should complete it, and we will grade it and provide a solution set for your edification.

PART B (Graded by Nick)

PROBLEM 1 (5+5+5+5 points)

Let L_1 be the language $\{a^n : n \geq 0\}$ and L_2 be the language $\{x : x \in \{a, b\}^* \text{ and } |x| = 5\}$.

(A) Which of the following sets are finite? Of those that are finite, what is their cardinality?

i. $L_1 \cap L_2$

ii. $L_2 \times (L_1 \cap L_2)$

iii. $L_2 \setminus L_1$.

(B) Which of the following sets contain the empty string ε?

i. $L_1 \cap L_2$

ii. $L_1 \cup L_2$

(C) Which of the following sets have the empty set $\emptyset$ as a subset?

i. L_2

ii. $L_1 \cap L_2$

(D) Which of the following sets contain $\emptyset$ as an element?

i. L_1

ii. $P(L_2)$

Solution.
(A) They all are.

i. 1.

ii. $2^5 = 32$.

iii. $2^5 - 1 = 31$.

(B) ii.
(C) The empty set is a subset of any set X, since tautologically, every element of $\emptyset$ is an element of X. So both i and ii include the empty set as a subset.
(D)

i. Obviously $\emptyset$ is not in L_1 unless we're representing strings in a very strange way.

ii. Since $\emptyset$ is a subset of L_2, $\emptyset \in \mathcal{P}(L_2)$.

PROBLEM 2 (Sipser 0.13 Challenge! 3 points)

An undirected graph is a set of points with lines connecting some of the points. The points are called nodes or vertices, and the lines are called edges. The number of edges at a particular node is the degree of that node. An edge from a node to itself is called a self-loop. (Sipser, pg. 10)

Show that every graph without self loops and with two or more nodes contains two nodes that have equal degrees.
(A) We begin with the following Observation: if S is a nonempty set of integers that are all greater than or equal to k, and if $|S| = n$, then S contains an element greater than or equal to $n + k - 1$. For if S does not include such an element, then S is a subset of the set $\{k, k + 1, \ldots, n + k - 2\}$ and therefore has fewer than n elements.

Now suppose G has n nodes, and for the sake of contradiction assume that no two nodes have the same degree. We claim that there is a node that is connected to every other node, or equivalently a node v of degree at least $n - 1$. This follows from our Observation: the set $\{\deg(v) : v \text{ is a node of } G\}$ is a set of n natural numbers that are all greater than or equal to 0, and therefore contains an element greater than or equal to $n - 1$. But now it follows (given that $n \geq 2$) that every node has degree at least 1 since v has degree $n - 1 \geq 1$ and every other node is connected to v. We can now use our observation again to show that there exists a node of degree at least n. This is a contradiction though, since there are only $n - 1$ other nodes it could possibly be connected to.