Harvard University
Computer Science 121

Problem Set 6

Due Tuesday, October 29, 2013 at 1:20 PM.
Submit your solutions electronically to cs121+ps6@seas.harvard.edu with "ps6 submission" in the subject line. The solutions to Parts A and B should be attached as separate PDF files, called lastname+ps6a.pdf and lastname+ps6b.pdf.
Late problem sets may be turned in until Friday, November 1, 2013 at 1:20 PM with a 20% penalty.

Problem set by **ENTER YOUR NAME HERE**

Collaboration Statement: **FILL IN YOUR COLLABORATION STATEMENT HERE (See the syllabus for information)**

See syllabus for collaboration policy.

PART B (Graded by Joy and Anupa)

Note: You may use implementation-level descriptions (Lecture 14, slide 14) for this part of the problem set unless the problem specifies otherwise.

PROBLEM 1 (10 points)

Define $\textsc{Prefix}(L) = \{x \mid xy \in L \text{ for some } y \in \Sigma^*\}$. Show that if L is Turing-recognizable, then $\textsc{Prefix}(L)$ is Turing-recognizable.

Solution 1: We use a recognizer M for L to construct a recognizer R for $\textsc{Prefix}(L)$: On input x, R will dovetail the computation of M on all strings xy, $y \in \Sigma^*$, accepting if M accepts any xy. This will accept if and only if x is a prefix of some string in L, so R recognizes $\textsc{Prefix}(L)$.
Solution 2: Since L is recognizable, there exists an enumerator E for L. We construct a recognizer R for $\textsc{Prefix}(L)$ as follows: On input x, R runs E, and for every string w E outputs, accepts if x is a prefix of w. This will accept if and only if x is a prefix of some string in L, so R recognizes $\textsc{Prefix}(L)$.
Solution 3: We construct an enumerator R for $\textsc{Prefix}(L)$ using an enumerator E for L: R: run E. For every string w output by E, output all prefixes of w. This will enumerate exactly the strings in $\textsc{Prefix}(L)$, which proves that $\textsc{Prefix}(L)$ is recognizable.

PROBLEM 2 (5 points)

A Modern Turing Machine (MTM), instead of $\{L, R\}$, has $\{L_k, R_k\}$ (move the head left or right k spaces on the tape, for any integer k encoded in the rule in the MTM). Prove or disprove: the MTMs are equivalent in power to the TMs.

True. Every TM is already an MTM in which $k = 1$ for every rule. To convert an MTM to an equivalent TM, replace every k-transition (where $k > 0$) with a set of k single transitions in the same direction, where all but the first read any tape symbol and do not alter it. Replace any 0-transitions with one right move (altering the letter as described in the 0-transition) and one left move (reading the symbol but not altering it). (It is important to move right, then left – not the other way around – since moving left may fail if we are at the left end of the tape.)

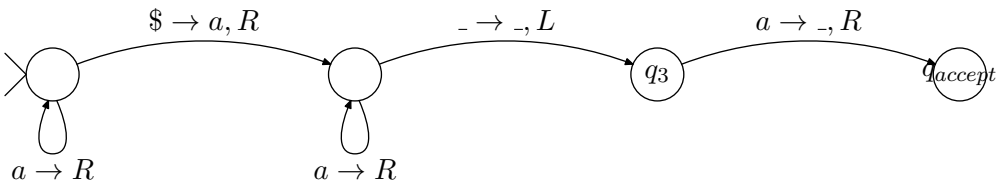
PROBLEM 3 (6+6 points)

A function $f : \Sigma^* \to \Sigma^*$ is *computable* if there is a Turing machine M that, on every input $w \in \Sigma^*$, halts (in state q_{accept}) with the tape containing just the string $f(w)$ (followed by blank symbols). In this case, we say that M *computes* the function f.

(A) Draw a state diagram for a (one-tape, deterministic) Turing machine M that performs unary addition. More specifically, it should take inputs of the form "$a^m\$a^n$", where m and n are positive integers, and output a^{m+n}. Your solution should not require more than four or five states. (For example, running M on input $aa\$a$ should result in aaa on the tape. You may assume that the input is well-formed.)

(B) Show that a function f over alphabet Σ is computable if and only if the language $L = \{w\$^i\sigma :$ the i'th symbol of $f(w)$ is $\sigma\}$ is decidable. (Thus, we can study computability of functions by studying decidability of languages.)

(A)



(B) Assume first that f is computable, with corresponding Turing Machine M_f. We will construct a Turing Machine M that decides L. Our machine will use two tapes, crucially relying on the fact proved in class that a k tape Turing Machine is equivalent in power to a one tape Turing Machine. The first tape will be the input tape. The machine will work as follows:

1. Scan input s on tape 1. If s is not of the form $w\$^i\sigma$, for $w \in \Sigma^*, \sigma \in \Sigma$, reject.
2. Copy the characters of s before the first \$ sign onto tape 2, i.e., copy w onto tape 2.
3. Simulate M_f on tape 2 with input w.
4. Move the tape head on tape 1 to the first dollar sign and the tape head on tape 2 to the first character of $f(w)$.
5. Advance both tape heads one symbol at a time to the right, crossing off characters until you reach σ on the first tape. If it matches the character under the tape head on tape 2, accept. Otherwise, reject.

This will reject any string not in L either in step 1 if it is of the wrong form, or in step 5 if the σ is not the i^{th} character of $f(w)$. The machine will accept all strings in L at step 5 because it checks for exactly the defining characteristics of a string in L.

Assume now that L is decidable with decider M. We will construct a Turing Machine M_f that computes f. This Turing Machine will have four tapes to minimize its conceptual difficulty. The first tape will be the input/output tape where w is given and $f(w)$ is returned. It works as follows:

1. Copy w onto the second tape.
2. Affix a single dollar sign to the end of the string on tape 2.
3. For each $\sigma \in \Sigma$, simulate on tape 3 the Turing Machine M with σ affixed to the end of the string on tape 2: i.e., compute $w\$^i\sigma$ on machine M.
4. If for any character σ, M acccepts the string, copy σ to the rightmost empty tile on tape 4 and repeat steps 2 through 4. If M rejects for all $\sigma \in \Sigma$, go to step 5
5. Copy the output of tape 4 to tape 1 and return. This is $f(w)$.

This machine works by taking w and using M to compute the characters of $f(w)$ on an individual basis. It simply checks all of the alphabet symbols to see which one is the i^{th} character of $f(w)$. It halts when there is no $\sigma \in \Sigma$ such that σ is the i^{th} character of $f(w)$, thus indicating $f(w)$ has no i^{th} character and thus that we must be at the end of $f(w)$.