Section 3 Handout

CS 121

September 26, 2013

Today's Topics

- Regular and Non-regular Languages
- Context-Free Grammars

1 Regular and Nonregular Languages

There are countably many regular expressions over a language, but there are uncountably many languages–so some of these languages *must not* be regular! But how do we find an explicitly non-regular language? We have two techniques: the pumping lemma and the closure properties of regular languages. You can use either of these techniques to prove (by contradiction) that a language is non-regular.

Pumping Lemma for regular languages:
If L is a regular language, then there exists a constant $p > 0$ such that for any string $w \in L$ with $|w| > p$, there exist strings x, y, $z \in \Sigma^*$, such that $w = xyz$, $|xy| \leq p$, $y \neq \epsilon$, and $xy^n z \in L$ for all $n \geq 0$.

Closure Properties:
Recall from lecture (and from last week's section) that regular languages are closed under *union, concatenation, Kleene Star, intersection, difference, complement, reversal.*

Exercise 1.1. *Which of the following are necessarily regular?*

- *A finite language.*
- *A union of finitely many regular languages.*
- $\{x : x \in L_1$ *and* $x \notin L_2\}$ *where* L_1 *and* L_2 *are regular.*
- *A superset of a regular language.*

1. Regular
2. Regular, by finite application of closure under union of 2 languages
3. Regular, by closure under complement, then intersection
4. Not necessarily regular ($\emptyset$ is regular)

Exercise 1.2. *Show that $L = \{a^i b^j : 0 \leq i < j\}$ is non-regular using the pumping lemma.*

Let p be the pumping length, and consider $a^p b^{p+1}$. $|xy| < p$, so y is all a's, so pumping y gives a string out of the language.

Exercise 1.3. *Let $L = \{ww | w \in \Sigma^*\}$. Show that L is non-regular using the pumping lemma.*

Let p be the pumping length, and consider $a^p b^p a^p b^p$. As above, y must be all a's, so pumping will decrease the number of b's on the left half, and increase it on the right, so the string is no longer in the language.

Exercise 1.4. *Let $L = \{w : w$ has more instances of substring aa than of substring $bb\}$ Show that L is nonregular.*

Intersect with $b^* a^*$, and this becomes language of strings $b^m a^n$ where $n > m$. Consider $b^p a^{p+1}$, and note that when y is pumped, it adds at least one b.

Exercise 1.5. *Show that $L = \{b^n c^{2^k} : n \geq 1, k \geq 1\}$ is non-regular.*

Proceed by showing the reversal is non-regular. Let p be the pumping length for the reversal, and consider $c^{2^p} b$. Let $|y|$ in the partitioning be m. After pumping, the new string is $c^{2^p + m} b$. So, it suffices to show that $2^p + m$ is not a power of 2. Since $|xy| < p$, $m < p$, and so $m < 2^p$. Since the next power of 2 after 2^p is $2^{p+1} = 2^p + 2^p$, $2^p + m$ is not a power of 2 as desired.

2 Context-Free Languages

Context-Free Grammars: A context-free grammar G is a four-tuple, defined as follows: $G = (V, \Sigma, R, S)$, where V (the set of variables) is an alphabet, Σ (the set of terminals) is a set disjoint from V, R is a finite set of rules, with each rule being a variable and a string of variables and terminals, and S (the start symbol) is an element of V.

Exercise 2.1. *Give a context-free grammar for $L = \{w : w$ is an even-length palindrome $\}$*

$S \to aSa | bSb | \epsilon$

Exercise 2.2. *(a) Give a context-free grammar for $L = \{w : w$ has three more a's than b's$\}$ over the alphabet $\Sigma = \{a, b\}$*

(a) Draw a parse tree for the string $baabaaa \in L$.

1. $S \to AAA$, $A \to bAA | AbA | AAb | a$
2. Derivation: $S \Rightarrow AAA \Rightarrow BAAAA \Rightarrow^* baaAA \Rightarrow^* baaBAAA \Rightarrow^* baabaaa$

Exercise 2.3. *Let $L = \{wy : w, y \in L(a^* \cup b^*)$ and $|w| = |y|\}$. Is L regular? Is L context-free?*

L is not regular since $L \setminus \{a\}^* \cup \{b\}^* = \{a^n b^n : n \in \mathbf{N}\}$. L is context free since $L = \{a^n b^n : n \in \mathbf{N}\} \cup \{aa\}^* \cup \{bb\}^*$.